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GENERALIZING DODGSON'S METHOD: A "DOUBLE-CROSSING" APPROACH TO COMPUTING DETERMINANTS

ABSTRACT. Dodgson's method of computing determinants was recently revisited in a paper that appeared in the *College Math Journal*. The method is attractive, but fails if an interior entry of an intermediate matrix has the value zero. This paper reviews the structure of Dodgson's method and introduces a generalization, called a "double-crossing" method, that provides a workaround to the failure for many interesting cases.

1. INTRODUCTION

Algebra students learn this simple pattern for the determinant of a 2×2 matrix:

$$\begin{vmatrix} a & & b \\ & & \\ c & & d \end{vmatrix} = ad - bc.$$

A similar pattern exists for 3×3 matrices, but to compute the determinant of larger matrices, the student must learn something a little more complicated. Most students learn to compute determinants by expansion of minors, first developed by Laplace. Some students also learn to compute determinants by triangularizing the matrix. Both methods are effective, and triangularization is efficient, but

- hand computations frequently lead to many mistakes;
- expansion of minors is tedious; and
- triangularization can turn a matrix of integers into a matrix of fractions.

In 1866, the Rev. Charles Lutwidge Dodgson[1], developed a conceptually simple method to compute determinants [2]. Dodgson's method iterates the familiar 2×2 formula; although it uses division, matrices with integer entries do not turn into matrices with rational entries.

Example 1. Given the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

we set $A^{(3)} = A$. To compute $A^{(2)}$, compute the determinants of the 2×2 contiguous submatrices of $A^{(3)}$:

$$A^{(2)} = \begin{pmatrix} \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} \\ & \\ \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 1 & 2 \end{pmatrix}.$$

To compute $A^{(1)}$, repeat the computation for $A^{(2)}$, but divide by the center entry of $A^{(3)}$:

$$A^{(1)} = \left(\frac{\begin{vmatrix} 3 & -3 \\ 1 & 2 \end{vmatrix}}{3} \right) = \left(\frac{9}{3} \right) = (3).$$

Now go back and compute $\det A$ using your favorite method (pattern, expansion of cofactors, triangularization, etc.). You will find that $\det A = 3$. $\diamondsuit$

Dodgson's method is *quick* and *conceptually simple*. In general, we can describe the method in the following way:

- Let $A^{(n)}$ be the $n \times n$ matrix given.

[1]Dodgson is better known to children of all ages as Lewis Carroll, author of "Alice in Wonderland" and "Jabberwocky".

- For each $k = n-1, n-2, \ldots, 1$:
  - Let $B^{(k)}$ be the $k \times k$ matrix of determinants of contiguous 2×2 submatrices of $A^{(k+1)}$.
  - If $k = n-1$, let $A^{(k)} = B^{(k)}$.
  - If $k \leq n-2$, let $A^{(k)}$ be the $k \times k$ matrix whose (i,j)-th element is the (i,j)-th element of $B^{(k)}$ divided by the (i,j)-th element of the interior of $A^{(k+2)}$. (The **interior** of an $n \times n$ matrix M is the $(n-1) \times (n-1)$ submatrix whose (i,j)-th element is the $(i+1, j+1)$-th element of M).
- The singleton element of $A^{(1)}$ is the determinant of A.

Dodgon's method is an example of a **condensation** method to compute determinants; each iterate $A^{(k)}$ is a condensation of the previous iterate $A^{(k+1)}$. Other condensation methods appear in [1]. If Dodgson's method terminates successfully, it computes the determinant of an $n \times n$ matrix using

$$4\left[(n-1)^2 + (n-2)^2 + \cdots + 1^2\right] \approx 4 \cdot n \cdot n^2$$

multiplications, subtractions, and divisions, or $O\left(n^3\right)$ operations in $\mathbb{Q}$. This is not bad, especially considering that all the divisions are exact, so no fractions are introduced; Bareiss' algorithm, a better-known fraction-free method of computing determinants, also performs $O\left(n^3\right)$ operations in $\mathbb{Q}$. [3, 4]

There's the rub, though: division presents Dodgson's method with a huge drawback. (It also presents an obstacle with the usage of Bareiss' algorithm.) What's so bad about division?

Example 2. Swap the first two rows of the matrix of Example 1 to obtain

$$M = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

We know from the properties of determinants that $\det M = -\det A = -3$.

What happens when we use Dodgson's method? First we compute

$$M^{(2)} = \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}.$$

To compute $M^{(1)}$ we divide the determinant of $M^{(2)}$ by the interior element of $M^{(3)}$. *But the interior element of $M^{(3)}$ is zero!* $\diamondsuit$

In general, Dodgson's method fails to compute the determinant of a matrix A whenever a zero appears in the interior of $A^{(k)}$ for any $k \geq 3$. This can happen even if no zeroes appear in the interior of A.

A workaround discussed in [2] swaps rows of the *original* matrix in such a way that zeroes are moved out of the interior. For example, if you swap the top two rows of M in Example 2, you return to A of example 1, for which Dodgson's method worked fine. However, there are two drawbacks to this workaround. *First,* swapping rows may well introduce other zeroes into the matrix, and it isn't easy to predict this from the outset. *Second,* swapping rows simply won't work for some matrices.

Example 3. No combination of row or column swaps will allow Dodgson's method to compute the determinant of

$$N = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix},$$

because there will *always* be a zero in the interior of N. $\diamondsuit$

Will a different workaround of Dodgson's method work for N? *Yes!* We describe such a method in Section 2. The reader will see quickly why we call it a "double-crossing" method.

2. A "DOUBLE-CROSSING" METHOD

Before describing the method, we illustrate it using the matrix from Example 2.

Example 4. Recall

$$M^{(3)} = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad M^{(2)} = \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}. \tag{2.1}$$

The zero in the interior of $M^{(3)}$ causes Dodgson's method to fail when computing $M^{(1)}$.

Above that zero is a non-zero element, $M^{(3)}_{1,2} = 3$. We will divide by this element instead, but this requires us to re-compute $M^{(2)}$ in a slightly different manner. Cross out the first row and second column of $M^{(3)}$ (the ones containing 3). We are left with the 2×2 complementary matrix

$$M^* = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Recall that we compute the *minor of an element* of a matrix by (again) crossing out the row and column containing that element, then taking the determinant of the remaining submatrix. Consider the matrix M' of *minor of elements* of $M^{(3)}$ that correspond to the elements of M^*; that is,

$$M' = \begin{pmatrix} \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} \\ \\ \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} \end{pmatrix}.$$

Put

$$M^{(2)} = M' = \begin{pmatrix} 2 & 1 \\ 3 & -3 \end{pmatrix}; \tag{2.2}$$

conclude by dividing $\det M^{(2)}$ by the non-zero element of $M^{(3)}$ that we identified earlier:

$$M^{(1)} = \left(\frac{\left| M^{(2)} \right|}{3} \right) = \left(\frac{-6-3}{3} \right) = \left(-\frac{9}{3} \right) = (-3).$$

As noted in example 2, $\det M = -3$.

Notice, by the way, that after a row and column swap all but one of the values in $M^{(2)}$ in line (2.2) are the values of $M^{(2)}$ in line (2.1). We will say more about this later. ◇

The generalization we have presented of Dodgson's method preserves its "spirit", inasmuch as we computed all determinants by condensing 2×2 contiguous submatrices. The example shows why we call it the "double-crossing" method:

- we found a non-zero element adjacent to the zero element;
- we crossed out its row and column, obtaining the complementary matrix M^*;
- for each element in M^*, we compute its minor by
  - (again) crossing out the row and column of its location in $M^{(3)}$, and
  - using the determinant of the remaining matrix to compute the element of $M^{(2)}$.

When a zero appears in an intermediate matrix $A^{(k)}$ (where $n > k \geq 3$), we cross out rows and columns that make up a *submatrix* of $A^{(n)}$. Let's look at a matrix where a zero does not appear in the interior of A, but does appear in the interior of an intermediate matrix.

Example 5. Let

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 5 & 3 & 1 & 0 \\ 1 & 3 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 2 & 0 & 2 & 0 & 1 \end{pmatrix}.$$

Using Dodgson's method, we compute $A^{(5)}$, $A^{(4)}$, and then

$$A^{(3)} = \begin{pmatrix} -4 & -2 & 2 \\ -2 & 0 & -2 \\ 4 & -4 & -1 \end{pmatrix}.$$

We encounter a zero in the interior! Above it is a non-zero element, -2. Notice that it lies in the top row and central column of $A^{(3)}$.

To compute $A^{(2)}$, do the following:

- cross out the top three rows and central three columns of $A^{(5)}$ (the *original* matrix);
- this gives us a 2×2 complementary matrix M^*;
- for each element in M^*,
  - cross out the row and column of its location in $A^{(5)}$,
  - compute the resulting minor, and
  - put it into the corresponding location in $A^{(2)}$.

Following these instructions, we have

$$A^* = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \qquad \text{and} \qquad A^{(2)} = \begin{pmatrix} \begin{vmatrix} 0 & 1 & 0 & 1 \\ 5 & 3 & 1 & 0 \\ 3 & 2 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 5 & 3 & 1 \\ 1 & 3 & 2 & 1 \\ 2 & 0 & 2 & 0 \end{vmatrix} \\ \\ \begin{vmatrix} 0 & 1 & 0 & 1 \\ 5 & 3 & 1 & 0 \\ 3 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 5 & 3 & 1 \\ 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix} \end{pmatrix}.$$

We compute the determinants for $A^{(2)}$ using Dodgson's method—and again, the bottom two values turn out to have values that you would obtain in the two top rows by ordinary condensation of $A^{(3)}$, while the others are different:

$$A^{(2)} = \begin{pmatrix} 2 & 0 \\ 4 & -4 \end{pmatrix}.$$

We now compute $A^{(1)}$ by dividing the determinant of $A^{(2)}$ by the non-zero entry of $A^{(3)}$ that we identified above: -2. We obtain

$$A^{(1)} = \left(\frac{\begin{vmatrix} 2 & 0 \\ 4 & -4 \end{vmatrix}}{-2} \right) = \left(\frac{-8}{-2} \right) = (4)\,,$$

and in fact $\det A = 4$. ◇

So far we've been using a special case of the double-crossing method. Theorem 6 describes this special case precisely.

Theorem 6 (Double-crossing method, special case)**.** *Let A be an $n \times n$ matrix. Suppose that we try to evaluate $\det A$ using Dodgson's method, but we encounter a zero in the interior of $A^{(k)}$, say in row i and column j of $A^{(k)}$. If the element α in row $i-1$ and column j of $A^{(k)}$ is non-zero, then we compute $A^{(k-1)}$and $A^{(k-2)}$ as usual, with the following exception for the element in row $i-1$ and column $j-1$ of $A^{(k-2)}$:*

- *let $\ell = n - k$;*
- *identify the $(\ell+3) \times (\ell+3)$ submatrix $\mathcal{A}$ whose upper left corner is the element in row $i-1$ and column $j-1$ of $A^{(n)}$;*
- *identify the 2×2 complementary matrix M^* by crossing out the $(\ell+1) \times (\ell+1)$ submatrix of $\mathcal{A}$ whose upper left corner is the element in row 1 and column 2 of $\mathcal{A}$;*
- *compute the matrix M' of determinants of minors of elements of M^* in $\mathcal{A}$;*
- *compute the element in row $i-1$ and column $j-1$ of $A^{(k-2)}$ by dividing the determinant of M' by α.*

We can use Dodgson's method to compute the intermediate determinants of M'. ◇

A proof of correctness appears in Section 4, after we explain why Dodgson's original method works correctly.

One can generalize Theorem 6 so that the non-zero element appears immediately above, below, left, right, or catty-corner to the zero; that is, the non-zero element is *adjacent* to the zero: see Theorem 13 on page 13. If the zero appears in a 3×3 block of zeroes, then the double-crossing method will not repair Dodgson's method, although additional strategies may be possible.

Before proceeding to the next section, we encourage the reader to go back and examine how Theorem 6 describes what we did in the examples of this section. We conclude by applying the double-crossing method to compute the determinant N from Example 3.

Example 7. Recall from Example 3

$$N = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}.$$

Put $N^{(4)} = N$; we have one zero element in the interior, at $(i, j) = (2, 3)$. As described in Theorem 6, this corresponds to element $(1, 2)$ of $N^{(2)}$. The other three elements of $N^{(2)}$ can be computed as usual:

$$N^{(3)} = \begin{pmatrix} -1 & 3 & 3 \\ 1 & -2 & -2 \\ 2 & -4 & 2 \end{pmatrix} \quad \longrightarrow \quad N^{(2)} = \begin{pmatrix} 1 & ? \\ 0 & -6 \end{pmatrix}.$$

We use the double-crossing method to compute the other element of $N^{(2)}$. The interior zero appears in the original matrix, so $\ell = 0$. We choose the 3×3 submatrix whose upper left corner is the element in row $i - 1 = 1$, column $j - 1 = 2$ of $N^{(4)}$:

$$\mathcal{A} = \begin{pmatrix} 0 & 3 & 0 \\ -1 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}.$$

Cross out the 1×1 submatrix in row 1, column 2 of $\mathcal{A}$ and identify the complementary matrix

$$M^* = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Compute the corresponding matrix of determinants of minors in $\mathcal{A}$

$$M' = \begin{pmatrix} \begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} \\ & \\ \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 3 & 3 \end{pmatrix}.$$

The determinant of this matrix is 9; after dividing by the nonzero $N^{(4)}_{1,3} = 3$ we have

$$N^{(2)} = \begin{pmatrix} 1 & 3 \\ 0 & -6 \end{pmatrix}.$$

We can now conclude by computing

$$N^{(1)} = \begin{pmatrix} \dfrac{\begin{vmatrix} 1 & 3 \\ 0 & -6 \end{vmatrix}}{-2} \end{pmatrix} = (3).$$

In fact, the determinant of N is 3. $\diamond$

The most computationally intensive part of the double-crossing method is that of computing the matrix of determinants of minors, but many of these *have already been computed*. It likely is not clear to the reader at present how to identify these, but as we explore the mechanics of Dodgson's method and the double-crossing method, we will see that we can predict exactly which parts of the matrix of cofactors need recomputing, and which can be copied from previous work.

In any case, it is time to consider why Dodgson's method works.

3. Why does Dodgson's method work?

Just as Bareiss' algorithm relies on a well-known theorem of Sylvester, Dodgson's method relies on the following theorem of Jacobi. [2]

Theorem (Jacobi's Theorem)**.** *Let*

- $\mathcal{A}$ *be an* $n \times n$ *matrix;*
- M *an* $m \times m$ *minor of* $\mathcal{A}$*, where* $m < n$*, chosen from rows* $i_1, i_2, \ldots, i_m$ *and columns* $j_1, j_2, \ldots, j_m$*,*
- M' *the corresponding* $m \times m$ *minor of* $\mathcal{A}'$*, the matrix of cofactors of* $\mathcal{A}$*, and*
- M^* *the complementary* $(n-m) \times (n-m)$ *minor of* $\mathcal{A}$*.*

Then

$$\det M' = (\det \mathcal{A})^{m-1} \cdot \det M^* \cdot (-1)^{\sum_{\ell=1}^{m} i_\ell + j_\ell} . \diamondsuit$$

Section 4 gives a proof of the new method.

We adopt the following notation. If $A \in \mathbb{R}^{m \times n}$ denotes a matrix, then A' denotes its matrix of cofactors. Both Dodgson's method and the double-crossing method make use of submatrices of a matrix, which for a given matrix A we denote in the following fashion:

$A_{i\ldots j,k\ldots\ell}$	rows $i, i+1, \ldots, j$ and columns $k, k+1, \ldots, \ell$ of A;
$A_{i,k\ldots\ell}$	row i and columns $k, k+1, \ldots, \ell$ of A;
$A_{i\ldots j,k}$	rows $i, i+1, \ldots, j$ and column k of A.

Example 8. Recall from Example 2

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Its matrix of cofactors is

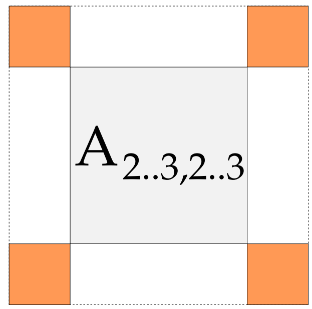
$$A' = \begin{pmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} \end{pmatrix}$$

while

$$A_{1\ldots 2,1\ldots 2} = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A_{3,2\ldots 3} = \begin{pmatrix} 1 & 1 \end{pmatrix} . \diamondsuit$$

To illustrate the relationship between Dodgson's method and Jacobi's Theorem, consider a generic 4×4 matrix.

FIGURE 3.1. Diagram illustrating how Jacobi's Theorem is applied to Dodgson's method, given the 4×4 matrix of Example 9.



Consider the generic 4×4 matrix from Example 9. The final condensation from Dodgson's method gave us

$$A^{(1)} = \left(\frac{|A_{1\ldots3,1\ldots3}||A_{2\ldots4,2\ldots4}| - |A_{2\ldots4,1\ldots3}||A_{1\ldots3,2\ldots4}|}{|A_{2\ldots3,2\ldots3}|} \right).$$

For Jacobi's Theorem, select the 2×2 minor of the corners of A

$$M = \begin{pmatrix} A_{1,1} & A_{1,4} \\ A_{4,1} & A_{4,4} \end{pmatrix}.$$

Its complementary minor is $M^* = (A_{2\ldots3,2\ldots3})$. The matrix of cofactors is

$$M' = \begin{pmatrix} A'_{1,1} & -A'_{1,4} \\ -A'_{4,1} & A'_{4,4} \end{pmatrix}$$

where $A'_{1,1} = |A_{2\ldots4,2\ldots4}|$, $A'_{4,4} = |A_{1\ldots3,1\ldots3}|$, $A'_{1,4} = |A_{2\ldots4,1\ldots3}|$, and $A'_{4,1} = |A_{1\ldots3,2\ldots4}|$. By Jacobi's Theorem,

$$\begin{aligned} \det M' &= (\det A)^{2-1} \cdot \det M^* \cdot (-1)^{1+1+4+4} \\ \frac{\det M'}{\det M^*} &= \det A \\ \frac{|A_{1\ldots3,1\ldots3}|\,|A_{2\ldots4,2\ldots4}| - |A_{2\ldots4,1\ldots3}|\,|A_{1\ldots3,2\ldots4}|}{|A_{2\ldots3,2\ldots3}|} &= \det A. \end{aligned}$$

This is precisely the final step in Dodgson's method as applied to A. Notice that the negatives in M' cancel when computing the determinant.

Example 9. Let A be a generic 4×4 matrix; we show how Dodgson's method applies Jacobi's Theorem. The first two condensations by Dodgson's method produce

$$A^{(3)} = \begin{pmatrix} |A_{1\ldots2,1\ldots2}| & |A_{1\ldots2,2\ldots3}| & |A_{1\ldots2,3\ldots4}| \\ |A_{2\ldots3,1\ldots2}| & |A_{2\ldots3,2\ldots3}| & |A_{2\ldots3,3\ldots4}| \\ |A_{3\ldots4,1\ldots2}| & |A_{3\ldots4,2\ldots3}| & |A_{3\ldots4,3\ldots4}| \end{pmatrix}$$

and

$$A^{(2)} = \begin{pmatrix} \dfrac{\begin{vmatrix} |A_{1\ldots2,1\ldots2}| & |A_{1\ldots2,2\ldots3}| \\ |A_{2\ldots3,1\ldots2}| & |A_{2\ldots3,2\ldots3}| \end{vmatrix}}{a_{2,2}} & \dfrac{\begin{vmatrix} |A_{1\ldots2,2\ldots3}| & |A_{1\ldots2,3\ldots4}| \\ |A_{2\ldots3,2\ldots3}| & |A_{2\ldots3,3\ldots4}| \end{vmatrix}}{a_{2,3}} \\ \dfrac{\begin{vmatrix} |A_{2\ldots3,1\ldots2}| & |A_{2\ldots3,2\ldots3}| \\ |A_{3\ldots4,1\ldots2}| & |A_{3\ldots4,2\ldots3}| \end{vmatrix}}{a_{3,2}} & \dfrac{\begin{vmatrix} |A_{2\ldots3,2\ldots3}| & |A_{2\ldots3,3\ldots4}| \\ |A_{3\ldots4,2\ldots3}| & |A_{3\ldots4,3\ldots4}| \end{vmatrix}}{a_{3,3}} \end{pmatrix}. \tag{3.1}$$

To see how $A^{(2)}$ corresponds to Jacobi's Theorem, consider the upper left 3×3 submatrix of $A^{(4)}$

$$\mathcal{A} = A_{1\ldots 3,1\ldots 3} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}.$$

Cross out row 2 and column 2 of $\mathcal{A}$, obtaining

$$M = \begin{pmatrix} a_{1,1} & a_{1,3} \\ a_{3,1} & a_{3,3} \end{pmatrix};$$

its complement in $\mathcal{A}$ is $M^* = (a_{2,2})$. The 2×2 submatrix of $\mathcal{A}'$ corresponding to the cofactors of M in $\mathcal{A}$ is

$$M' = \begin{pmatrix} \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} & -\begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix} \\ \\ -\begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix} & \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} \end{pmatrix} = \begin{pmatrix} |A_{2\ldots 3,2\ldots 3}| & -|A_{2\ldots 3,1\ldots 2}| \\ -|A_{1\ldots 2,2\ldots 3}| & |A_{1\ldots 2,1\ldots 2}| \end{pmatrix}.$$

Use Jacobi's Theorem and a column and row swap to obtain

$$\begin{aligned} \det M' &= (\det \mathcal{A})^{2-1} \cdot \det M^* \cdot (-1)^{1+1+3+3} \\ \frac{\det M'}{\det M^*} &= \det \mathcal{A} \\ \frac{\begin{vmatrix} |A_{1\ldots 2,1\ldots 2}| & -|A_{1\ldots 2,2\ldots 3}| \\ -|A_{2\ldots 3,1\ldots 2}| & |A_{2\ldots 3,2\ldots 3}| \end{vmatrix}}{a_{2,2}} &= \det \mathcal{A}. \end{aligned}$$

Since $\mathcal{A} = A_{1\ldots 3,1\ldots 3}$, we have computed the determinant of the upper 3×3 submatrix of A. This is equivalent to the element in the upper left corner of $A^{(2)}$ in 3.1; the negatives from the cofactors in Jacobi's method cancel each other out. Likewise,

- the upper right corner of $A^{(2)}$ has the value $\det A_{1\ldots 3,2\ldots 4}$;
- the lower left corner has the value $\det A_{2\ldots 4,1\ldots 3}$; and
- the lower right corner has the value $\det A_{2\ldots 4,2\ldots 4}$.

The next (and final) condensation in Dodgson's method is

$$A^{(1)} = \left(\frac{|A^{(2)}|}{A^{(3)}_{2,2}} \right) = \left(\frac{|A_{1\ldots 3,1\ldots 3}||A_{2\ldots 4,2\ldots 4}| - |A_{2\ldots 4,1\ldots 3}||A_{1\ldots 3,2\ldots 4}|}{|A_{2\ldots 3,2\ldots 3}|} \right).$$

Apply Jacobi's Theorem *on* A, using for M^* the 2×2 interior submatrix of A, and we have

$$|A| = \frac{|A_{1\ldots 3,1\ldots 3}||A_{2\ldots 4,2\ldots 4}| - |A_{2\ldots 4,1\ldots 3}||A_{1\ldots 3,2\ldots 4}|}{|A_{2\ldots 3,2\ldots 3}|}.$$

This is precisely the singleton element of $A^{(1)}$! See Figure 3.1. ◇

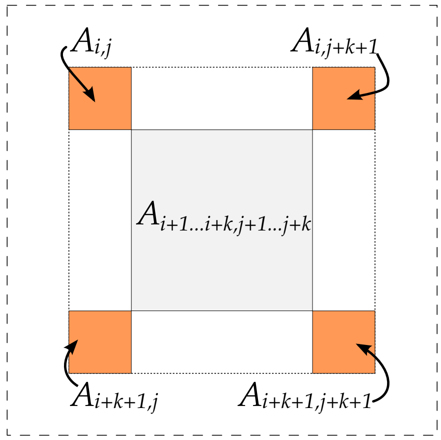
Theorem 10. *Let A be an $n \times n$ matrix. After k successful condensations, Dodgson's method produces the matrix*

$$A^{(n-k)} = \begin{pmatrix} |A_{1\ldots k+1,1\ldots k+1}| & |A_{1\ldots k+1,2\ldots k+2}| & \cdots & |A_{1\ldots k+1,n-k\ldots n}| \\ |A_{2\ldots k+2,1\ldots k+1}| & |A_{2\ldots k+2,2\ldots k+2}| & \cdots & |A_{2\ldots k+2,n-k\ldots n}| \\ \vdots & \vdots & \ddots & \vdots \\ |A_{n-k\ldots n,1\ldots k+1}| & |A_{n-k\ldots n,2\ldots k+2}| & \cdots & |A_{n-k\ldots n,n-k\ldots n}| \end{pmatrix}$$

whose entries are the determinants of the $(k+1) \times (k+1)$ submatrices of A. ◇

By "successful" iterations we mean that one never encounters division by zero.

FIGURE 3.2. Diagram illustrating how Jacobi's Theorem is applied to Dodgson's method, given a general $n \times n$ matrix.



Consider a $(k+1) \times (k+1)$ submatrix of A, $\mathcal{A} = A_{i...i+k+1,j...j+k+1}$. Select the 2×2 minor

$$M = \begin{pmatrix} A_{i,j} & A_{i,j+k+1} \\ A_{i+k+1,j} & A_{i+k+1,j+k+1} \end{pmatrix}.$$

Its complementary minor is $M^* = \left(A_{i+1...i+k,j+1...j+k} \right)$. The matrix of cofactors of M in $\mathcal{A}$ is

$$M' = \begin{pmatrix} A'_{i,j} & \pm A'_{i,j+k+1} \\ \pm A'_{i+k+1,j} & A'_{i+k+1,j+k+1} \end{pmatrix}$$

where

$$\begin{aligned} A'_{i,j} &= \left| A_{i+1...i+k+1,j+1...j+k+1} \right| \\ A'_{i,j+k+1} &= \left| A_{i+1...i+k+1,j...j+k} \right| \\ A'_{i+k+1,j} &= \left| A_{i...i+k,j+1...j+k+1} \right| \\ A'_{i+k+1,j+k+1} &= \left| A_{i...i+k,j...j+k} \right|. \end{aligned}$$

By Jacobi's Theorem,

$$\det M' = (\det \mathcal{A})^{2-1} \cdot \det M^* \cdot (-1)^{i+(j+k+1)+j+(i+k+1)}$$

or

$$\det \mathcal{M} = \frac{\left| A'_{i,j} \right| \cdot \left| A'_{i+k+1,j+k+1} \right| - \left| A'_{i,j+k+1} \right| \cdot \left| A'_{i+k+1,j} \right|}{\left| A_{i+1...i+k,j+1...j+k} \right|}, \tag{3.2}$$

so long as the denominator is not zero.

Proof. We proceed by induction on k.

Inductive Base: When $k=1$, the theorem is trivial: one condensation gives

$$A^{(n-1)} = \begin{pmatrix} |A_{1\ldots 2,1\ldots 2}| & |A_{1\ldots 2,2\ldots 3}| & \cdots & |A_{1\ldots 2,n-1\ldots n}| \\ |A_{2\ldots 3,1\ldots 2}| & |A_{2\ldots i+2,2\ldots i+2}| & \cdots & |A_{2\ldots 3,n-1\ldots n}| \\ \vdots & \vdots & \ddots & \vdots \\ |A_{n-1\ldots n,1\ldots 2}| & |A_{n-i\ldots n,2\ldots i+2}| & \cdots & |A_{n-1\ldots n,n-1\ldots n}| \end{pmatrix}.$$

Inductive Hypothesis: Fix k. Assume that for all $\ell = 1,\ldots,k$, the ℓth condensation gives us $A^{(n-\ell)}$ where for all $1 \le i,j \le n$

$$A^{(n-\ell)}_{i,j} = \left|A_{i\ldots i+(n-\ell),j\ldots j+(n-\ell)}\right|.$$

Inductive Step: Let $i,j \in \{1,\ldots,n-k\}$. The next condensation in Dodgson's method gives us

$$A^{(k+1)}_{i,j} = \frac{A^{(k)}_{i,j}A^{(k)}_{i+1,j+1} - A^{(k)}_{i+1,j}A^{(k)}_{i,j+1}}{A^{(k-1)}_{i+1,j+1}}.$$

From the inductive hypothesis, we can substitute

$$A^{(k+1)}_{i,j} = \frac{\left(\begin{array}{l} \left|A_{i\ldots i+k,j\ldots j+k}\right| \left|A_{i+1\ldots i+k+1,j+1\ldots j+k+1}\right| \\ \qquad - \left|A_{i+1\ldots i+k+1,j\ldots j+k}\right| \left|A_{i\ldots i+k,j+1\ldots j+k+1}\right| \end{array}\right)}{\left|A_{i+1\ldots i+k,j+1\ldots j+k}\right|}.$$

Apply Jacobi's Theorem with

- $\mathcal{A} = A_{i\ldots i+k+1,j\ldots j+k+1}$,
- M the 2×2 minor made up of the corners of $\mathcal{A}$,
- M' the corresponding 2×2 minor of $\mathcal{A}'$, and
- M^* the complementary $k\times k$ minor of $\mathcal{A}$

to see that

$$A^{(k+1)}_{i,j} = \left|A_{i\ldots i+k+1,j\ldots j+k+1}\right|.$$

(See Figure 3.2.) □

4. WHY DOES THE "DOUBLE-CROSSING" METHOD WORK?

The new workaround is based on Theorem 10. The goal in step i of the algorithm is to compute each determinant $\left|A_{i\ldots i+k,j\ldots j+k}\right|$. Dodgson's method fails when the corresponding denominator of (3.2) is zero.

However, the fraction of (3.2) is not the only way to apply Jacobi's Theorem. As long as *some* $(i-1)\times(i-1)$ minor of $A_{i\ldots i+k,j\ldots j+k}$ has non-zero determinant, we can still recover, reusing most of the computations already performed in previous steps of Dodgson's method, and calculating only a few new minors, again using the same approach as Dodgson's method. For example, the proof of Theorem 6 can be summarized by applying Jacobi's Theorem with

- $\mathcal{A}$ of Theorem 6 standing in for $\mathcal{A}$ of Jacobi's Theorem;
- the 2×2 submatrix M^* of Theorem 6 standing in for M^* of Jacobi's Theorem; and
- the matrix of determinants of minors of M' of Theorem 6 standing in for M' of Jacobi's Theorem.

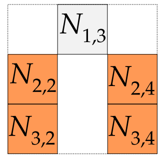
One difference does require investigation: the negatives in the matrix of cofactors are now in different places! With Dodgson's Method, we choose the central minor for M, which has two important consequences in Jacobi's Theorem:

- $i_\ell = j_\ell$ for each ℓ, so that $(-1)^{\sum_{\ell=1}^m i_\ell + j_\ell} = (-1)^{\sum_{\ell=1}^m 2i} = 1$; and
- if negatives appear in the 2×2 matrix of cofactors, they appear off the main diagonal, cancelling each other out.

In the special case of the double-crossing method, the matter is a little more complicated. We have chosen the top-middle minor for M, so:

- $(-1)^{\sum_{\ell=1}^m i_\ell + j_\ell} = (-1)^{1+2+2+3+\cdots+k+k+1} = (-1)^{1+k}$; and

FIGURE 4.1. Diagram for workaround for Example 11



The problem in computing $N^{(2)}_{1,2}$ was the zero in position $N_{2,3}$. Since $N^{(2)}_{1,2} = \det N_{2\dots,4,2\dots4}$ choose instead the minor

$$M = \begin{pmatrix} N_{2,2} & N_{2,4} \\ N_{3,2} & N_{3,4} \end{pmatrix}$$

whose complementary minor is $M^* = N_{1,3} = 3$. In this case the matrix of cofactors is

$$M' = \begin{pmatrix} (-1)^{2+1}\begin{vmatrix} N_{1,3} & N_{1,4} \\ N_{3,3} & N_{3,4} \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} N_{1,2} & N_{1,3} \\ N_{3,2} & N_{3,3} \end{vmatrix} \\ \\ (-1)^{3+1}\begin{vmatrix} N_{1,3} & N_{1,4} \\ N_{2,3} & N_{2,4} \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} N_{1,2} & N_{1,3} \\ N_{2,2} & N_{2,3} \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -\begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} \\ \\ \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 3 & 3 \end{pmatrix}.$$

By Jacobi's Theorem,

$$\det N_{2\dots,4,2\dots4} = \frac{\det M'}{\det M^* \cdot (-1)^{1+2+3+3}} = \frac{-9}{-3} = 3.$$

Notice the correspondence between M' and the matrix $N^{(2)}$ obtained in Example 7 using the double-crossing method: multiplying the top row of M' by -1 gives us $N^{(2)}$. Dividing $\det M'$ by $-3 = 3 \cdot (-1)^9$ cancels out the negative introduced into the determinant.

- the determinant of the 2×2 matrix of cofactors is
$$(-1)^{2i+2j+3k-1} \cdot D$$
where D is the determinant of the minors. The multiple of D simplifies to $(-1)^{k-1}$, which has the same sign as $(-1)^{1+k}$; thus we can disregard the signs and consider only the determinants of the minors.

We use this to revisit Example 3, which would not work with Dodgson's method even after swapping rows or columns.

Example 11. Recall from Example 3

$$N = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}.$$

Dodgson's method would have us compute $N^{(2)}_{1,2}$ by choosing for Jacobi's Theorem

$$M = \begin{pmatrix} N_{1,2} & N_{1,4} \\ N_{3,2} & N_{3,4} \end{pmatrix} \qquad \Longrightarrow \qquad M^* = (N_{2,3}),$$

whence

$$N^{(2)}_{1,2} = \det N_{1\dots3,2\dots4} = \frac{\det M'}{\det M^*}$$

but $\det M^* = N_{2,3} = 0$.

The double-crossing method allows us to choose instead

$$M = \begin{pmatrix} N_{2,2} & N_{2,4} \\ N_{3,2} & N_{3,4} \end{pmatrix} \qquad \Longrightarrow \qquad M^* = (N_{1,3}).$$

Since $N_{1,3} = 1$ we have

$$\begin{aligned}
N_{1,2}^{(2)} &= \det N_{1\ldots3,2\ldots4} \\
&= \frac{\det M'}{\det M^* \cdot (-1)^{1+2+3+3}} \\
&= \frac{\det M'_{1,1} \cdot \det M'_{2,2} - \det M'_{1,2} \cdot \det M'_{2,1}}{-N_{1,3}} \\
&= \frac{-\left(\det M'_{1,1} \cdot \det M'_{2,2} - \det M'_{1,2} \cdot \det M'_{2,1}\right)}{-N_{1,3}} \\
&= \frac{(-1)^{2+1}\begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix} \cdot (-1)^{3+3}\begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix} - (-1)^{2+3}\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} \cdot (-1)^{3+1}\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix}}{-3} \\
&= \frac{-9}{-3} = 3.
\end{aligned}$$

See Figure 4.1. ◇

We conclude by showing how Jacobi's Theorem likewise justifies the "double-crossing" method in Example 5.

Example 12. Recall from Example 5

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 5 & 3 & 1 & 0 \\ 1 & 3 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 2 & 0 & 2 & 0 & 1 \end{pmatrix}.$$

When computing $A^{(1)}$, Dodgson's method wants us to cross out the middle three rows and the middle three columns of A, obtaining the minor

$$M = \begin{pmatrix} A_{1,1} & A_{1,5} \\ A_{5,1} & A_{5,5} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

whose complementary minor is

$$M^* = \begin{pmatrix} A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix} = \begin{pmatrix} 5 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Unfortunately, the determinant of this latter matrix is zero, which corresponds to the zero in the interior of

$$A^{(3)} = \begin{pmatrix} -4 & -2 & 2 \\ -2 & 0 & -2 \\ 4 & -4 & 3 \end{pmatrix}.$$

The double-crossing method suggests instead to cross out the top three rows and middle three columns of A, obtaining the minor

$$M = \begin{pmatrix} A_{4,1} & A_{4,5} \\ A_{5,1} & A_{5,5} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}$$

whose complementary minor is

$$M^* = \begin{pmatrix} A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,2} & A_{3,3} & A_{3,4} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 5 & 3 & 1 \\ 3 & 2 & 1 \end{pmatrix}.$$

The determinant of M^* is $A_{1,2}^{(3)} = -2$; we will multiply it by $(-1)^{1+2+2+3+3+4} = -1$. To compute $A_{1,2}^{(2)}$ we need $\det M'$, which requires us to compute the matrix of cofactors of M in $A^{(5)}$:

$$M' = \begin{pmatrix} -\begin{vmatrix} 0 & 1 & 0 & 1 \\ 5 & 3 & 1 & 0 \\ 3 & 2 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 5 & 3 & 1 \\ 1 & 3 & 2 & 1 \\ 2 & 0 & 2 & 0 \end{vmatrix} \\ \\ \begin{vmatrix} 0 & 1 & 0 & 1 \\ 5 & 3 & 1 & 0 \\ 3 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 5 & 3 & 1 \\ 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix} \end{pmatrix}.$$

The top row of negatives introduces a negative into $\det M'$, which cancels with the -1 that is multiplied to $\det M^*$, cancelling each other out. We can thus consider the determinants of minors, rather than the matrix of cofactors. ◊

Of course, it is possible that both an element of the interior is zero *and* the element above it is zero. The following theorem provides the promised generalization of Theorem 6; its proof uses Jacobi's method in a manner similar to the proof of Theorem 6. Notice that if $r = s = 0$, then Theorem 13 specializes to Dodgson's method.

Theorem 13. *Let A be an $n \times n$ matrix. Suppose that we try to evaluate $|A|$ using Dodgson's method, but we encounter a zero in the interior of $A^{(k)}$, say in row i and column j of $A^{(k)}$. Let $r, s \in \{-1, 0, 1\}$. If the element α in row $i + r$ and column $j + s$ of $A^{(k)}$ is non-zero, then we compute $A^{(k-2)}$ as usual, with the following exception for the element in row $i - 1$ and column $j - 1$:*

- *let $\ell = n - k$;*
- *identify the $(\ell + 3) \times (\ell + 3)$ submatrix $\mathcal{A}$ whose upper left corner is the element in row $i - \ell$ and column $j - \ell$ of $A^{(n)}$;*
- *identify the complementary minor M^* by crossing out the $(\ell + 1) \times (\ell + 1)$ submatrix M of $\mathcal{A}$ whose upper left corner is the element in row $r + 2$ and column $s + 2$ of $\mathcal{A}$;*
- *compute the matrix M' of determinants of minors of M^* in $\mathcal{A}$;*
- *compute the element in row $i - 1$ and column $j - 1$ of $A^{(k+2)}$ by dividing the determinant of M' by α.*

We can use Dodgson's method to compute the determinants of M'. ◊

5. Concluding remarks

The double-crossing method succeeds as long as Dodgson's method succeeds, and with the same number of integer operations, since Dodgson's method is a special case of the double-crossing method. In the worst case scenario where Dodgson's method generates a number of interior zeroes, the double-crossing method requires the computation of at least two $(k-1) \times (k-1)$ determinants for every zero entry of $A^{(k)}$. However, this changes the number of integer operations only by a constant, so the double-crossing method requires only $O(n^3)$ integer operations. In addition, the double-crossing method preserves the general simplicity and spirit of Dodgson's method.

The double-crossing method is not guaranteed to succeed; if an intermediate matrix contains a 3×3 block of zeroes, then the double-crossing method also fails. Sparse matrices provide an excellent example where the double-crossing method is an abject failure; consider the identity matrix of order six or higher.

In those cases where the double-crossing method fails, one can still preserve the computations that work, and adapt a hybrid with another method. Theorem 13 tells us the precise minor $\mathcal{A}$ whose determinant we need; we can compute the determinant of this minor using another method, substitute its value into row $i - 1$ and column $j - 1$ of $A^{(k+2)}$, and proceed.

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